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LETTER TO THE EDITOR

Gauge-invariant Lagrangian wave equations of arbitrary helicity

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Abstract. The non-trivially gauge-invariant Lagrangian field equations of arbitrary helicity have been constructed by integration of the trivially gauge-invariant non-Lagrangian Poincaré field strengths based on Lorentz irreps of unmixed spin. The details of the gauge properties of the Lagrangian fields are obtained systematically, for arbitrary spin, from the very simple field-strength formulation as a consequence of the integration.

The study of the Lagrangian formulation for arbitrary spin was started in 1939 by Fierz and Pauli [1] and in 1978 Fronsdal and Fang [2, 3] obtained the massless Lagrangian fields for arbitrary spin. In 1980 de Wit and Freedman [4] used an elegant hierarchy of generalised Christoffel symbols with simple gauge properties to demonstrate the systematics of higher-spin gauge fields. A feature common to these analyses of higher-spin theories has been the *a priori* assumption of some of the gauge properties of the spin-j Lagrangian potentials. These forms have then been validated by demonstrating that the equations to which they apply are indeed those of Poincaré irreps of spin j.

In this letter we report that such assumptions are unnecessary and that the form of the gauge-invariant classical Lagrangian fields of arbitrary helicity, including all the details of the gauge properties, may be obtained from the very simple field-strength formulation of the Poincaré group irreps by an essentially algebraic analysis.

Our method, in essence, reverses the analyses of de Wit and Freedman [4], Burgers [5] and Berends *et al* [6] who began with the Lagrangian potentials and defined the field strengths in terms of derivatives of these. We, however, define the field strengths of arbitrary helicity directly in terms of the unmixed spin Lorentz irreps $(j, 0)\oplus(0, j)$ and systematically integrate them to obtain correctly constrained Lagrangian potentials of Fierz-Pauli [1] and Rarita-Schwinger [7] types corresponding to the irreps $(\frac{1}{2}j, \frac{1}{2}j)$ and $(\frac{1}{2}(n+1), \frac{1}{2}n)\oplus(\frac{1}{2}n, \frac{1}{2}(n+1))$ respectively, where $n = j-\frac{1}{2}$ for the fermionic case. The analysis supplies the appropriate field equations and in particular the non-trivial gauge invariance of the Lagrangian potential formulation arises naturally from the arbitrariness necessarily associated with each integration.

Individual lower-spin cases have been dealt with previously by Doughty, Wiltshire and Collins [8-11]. The method presented here is essentially uniform for all values of spin. We begin with the direct sum of two Weyl spinors of types $(j, 0)$ and $(0, j)$ which we write as a totally symmetric $2(2j+1)$-component spin-j Dirac multispinor

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$\psi = (\psi_{\alpha_1 \ldots \alpha_{2j}})$ satisfying $(\gamma_5 \psi)_{\alpha_1 \ldots \alpha_{2j}} = (\gamma_5 \psi)_{(\alpha_1 \ldots \alpha_{2j})}$ and obeying the very simple massless Bargmann-Wigner equation [12], $\partial\!\!\!/ \psi = 0$. From ψ we define [8-11] a field strength F by

$$F_{\mu_1 \nu_1 \ldots \mu_n \nu_n} = \frac{1}{(4)^n} \mathrm{Tr}(\ldots \mathrm{Tr}(\psi C^{-1} \gamma_{\mu_1 \nu_1}) \ldots C^{-1} \gamma_{\mu_n \nu_n}) \tag{1}$$

where C is the charge conjugation matrix and $\gamma_{\mu\nu} = \frac{1}{2}\mathrm{i}(\gamma_\mu \gamma_\nu - \gamma_\nu \gamma_\mu)$ is the generator of Lorentz transformations in the Dirac irrep. F is a tensor for integer spin $j = n$ and is a tensor-spinor with one free (suppressed) Dirac index for half-integer spin $j = n + \frac{1}{2}$. F contains the same information as ψ due to the inverse relation

$$\psi = \frac{1}{(2)^n} F_{\mu_1 \nu_1 \ldots \mu_n \nu_n} (\gamma^{\mu_1 \nu_1} C \otimes \cdots \otimes \gamma^{\mu_n \nu_n} C). \tag{2}$$

F has the same symmetries as the field strength R of Berends *et al* [6]. In particular, it obeys the equations [6, 13]

$$F^{\lambda}{}_{\nu_1 \lambda \nu_2 \ldots \mu_n \nu_n} = 0 \tag{3}$$

$$F_{\mu_1 \nu_1 \ldots \mu_{n-1} \nu_{n-1} [\mu_n \nu_n, \lambda]} = 0 \tag{4}$$

$$\partial^{\lambda} F_{\lambda \nu_1 \ldots \mu_n \nu_n} = 0. \tag{5}$$

For the fermionic case, F also satisfies

$$\gamma^{\lambda} F_{\lambda \nu_1 \ldots \mu_n \nu_n} = 0 \tag{6}$$

$$\partial\!\!\!/ F_{\mu_1 \nu_1 \ldots \mu_n \nu_n} = 0. \tag{7}$$

In both cases the above equations have no gauge freedom. We define an associated spin-j field strength $\hat{F} \equiv \Gamma^{(n)}$ by

$$\hat{F}^{\rho_1 \rho_2 \ldots \rho_n}{}_{\mu_1 \mu_2 \ldots \mu_n} = F^{\rho_1}{}_{(\mu_1}{}^{\rho_2 \ldots}{}_{\mu_2 \ldots}{}^{\rho_n}{}_{\mu_n)} \tag{8}$$

which has identical symmetries to the generalised Riemann tensor, $R \equiv \Gamma^{(s)}$, of de Wit and Freedman [4]. F may be recovered from $\hat{F}$ by

$$F_{\rho_1 \mu_1 \rho_2 \mu_2 \ldots \rho_n \mu_n} = \frac{2^n}{n+1} \hat{F}_{\rho_1 [\rho_2 \ldots [\rho_n}{}^{\mu_1 \mu_2] \ldots \mu_n]} \tag{9}$$

showing that each contains the same information.

By equation (4) the Poincaré lemma [14] guarantees the existence of a field A satisfying

$$F_{\rho_1 \mu_1 \ldots \rho_n \mu_n} = A_{\rho_1 \ldots \rho_{n-1} \mu_1 \ldots \mu_{n-1} [\mu_n, \rho_n]} \tag{10}$$

and free to be varied by

$$\delta A_{\rho_1 \ldots \rho_{n-1} \mu_1 \ldots \mu_{n-1} \mu_n} = B_{\rho_1 \ldots \rho_{n-1} \mu_1 \ldots \mu_{n-1}, \mu_n} \tag{11}$$

where B is arbitrary. This freedom can be used to construct [15] a highly symmetric field $\Gamma^{(n-1)}$ from A (obeying symmetries (19)-(21) below) in much the same way that conditions may be imposed on gauge fields by gauge fixing. We obtain

$$\hat{F}_{\rho_1 \ldots \rho_n \mu_1 \ldots \mu_n} = \Gamma^{(n-1)}{}_{\rho_1 \ldots \rho_{n-1} \mu_1 \ldots \mu_n, \rho_n} - \Gamma^{(n-1)}{}_{\rho_1 \ldots \rho_n (\mu_2 \ldots \mu_n, \mu_1)} \tag{12}$$

and the most general variation of $\Gamma^{(n-1)}$ which preserves its symmetries and leaves $\hat{F}$ invariant is

$$\delta\Gamma^{(n-1)}{}_{\rho_1\cdots\rho_{n-1}\mu_1\cdots\mu_n} = \varepsilon_{\rho_1\cdots\rho_{n-1},\mu_1\cdots\mu_n} \tag{13}$$

with ε satisfying

$$\varepsilon_{\rho_1\cdots\rho_{n-1}} = \varepsilon_{(\rho_1\cdots\rho_{n-1})} \tag{14}$$

$$\varepsilon^{\lambda}{}_{\lambda\rho_3\cdots\rho_{n-1}} = 0 \qquad (n \geqslant 3) \tag{15}$$

$$\gamma^{\lambda}\varepsilon_{\lambda\rho_2\cdots\rho_{n-1}} = 0 \qquad (\text{fermionic}, j \geqslant \tfrac{5}{2}). \tag{16}$$

The symmetries of $\hat{F}$ imply

$$\Gamma^{(n-1)}{}_{\rho_1\cdots\rho_{n-1}[\rho_{n-1}\mu_1}{}^{\mu_2\cdots\mu_n,\lambda]} = 0 \tag{17}$$

and a sequence of essentially identical 'integrations' using the Poincaré lemma reproduces [15] the hierarchy of generalised Christoffel symbols of de Wit and Freedman [4], namely

$$\Gamma^{(m)}{}_{\rho_1\cdots\rho_m\mu_1\cdots\mu_n} = \Gamma^{(m-1)}{}_{\rho_1\cdots\rho_{m-1}\mu_1\cdots\mu_n,\rho_m} - \frac{n}{m}\Gamma^{(m-1)}{}_{\rho_1\cdots\rho_m(\mu_2\cdots\mu_n,\mu_1)} \tag{18}$$

$$\Gamma^{(m)}{}_{\rho_1\cdots\rho_m\mu_1\cdots\mu_n} = \Gamma^{(m)}{}_{(\rho_1\cdots\rho_m)(\mu_1\cdots\mu_n)} \tag{19}$$

$$\Gamma^{(m)\lambda}{}_{\lambda\rho_3\cdots\rho_m\mu_1\cdots\mu_n} = 0 \qquad (m, n \geqslant 2) \tag{20}$$

$$\gamma^{\lambda}\Gamma^{(m)}{}_{\lambda\rho_2\cdots\rho_m\mu_1\cdots\mu_n} = 0 \qquad (\text{fermionic } m, n \geqslant 1). \tag{21}$$

Each integration initially produces a field with substantial gauge freedom, as with A in equations (10) and (11). However, unlike $\Gamma^{(n-1)}$, constructing each $\Gamma^{(m)}$ ($m \leqslant n-2$) to satisfy equations (18)-(21) totally exhausts this freedom. That is, for a given $m \leqslant n-2$ no variation of $\Gamma^{(m)}$ preserves all of equations (18)-(21) (we exclude the trivial addition of a constant field). Nevertheless, the freedom (13) allows the self-consistent variations

$$\delta\Gamma^{(m)}{}_{\rho_1\cdots\rho_m\mu_1\cdots\mu_n} = a_m\varepsilon_{\rho_1\cdots\rho_m(\mu_{m+2}\cdots\mu_n,\mu_1\cdots\mu_{m+1})} \tag{22}$$

where $a_m = (-1)^{n-m-1}(n-1)!/[(n-m-1)!m!]$ and ε obeys equations (14)-(16). Applied to all the $\Gamma^{(m)}$ simultaneously, this variation preserves equations (18)-(21), leaves $\hat{F}$ invariant and is precisely the usual arbitrary spin gauge freedom.

The result of carrying out all the above steps is to establish the existence of the completely symmetric potential $\Gamma^{(0)}_{\mu_1\cdots\mu_n}$ related to the field strength by equation (18). For integer spin $n \geqslant 2$, the trace condition (20) on $\Gamma^{(2)}$ is the field equation [4] for $\phi \equiv \Gamma^{(0)}$, namely

$$\Box\phi_{\mu_1\cdots\mu_n} - n\partial^{\lambda}\partial_{(\mu_1}\phi_{\mu_2\cdots\mu_n)\lambda} + \tfrac{1}{2}n(n-1)\partial_{(\mu_1}\partial_{\mu_2}\phi_{\mu_3\cdots\mu_n)}{}^{\lambda}{}_{\lambda} = 0. \tag{23}$$

For fermionic spin $j \geqslant \frac{3}{2}$, the γ trace (21) on $\Gamma^{(1)}$ constitutes the field equation for $\psi \equiv \Gamma^{(0)}$:

$$\not{\partial}\psi_{\mu_1\cdots\mu_n} - n\partial_{(\mu_1}\gamma^{\lambda}\psi_{\mu_2\cdots\mu_n)\lambda} = 0. \tag{24}$$

(The spin-1 field equation corresponds to equation (5) and for spin-$\frac{1}{2}$ the Bargmann–Wigner equation is itself the field equation.) Although these forms of the field equations are not directly derivable from a Lagrangian they may be combined with their own traces to give the equivalent standard spin-j Lagrangian equations [4, 5, 9, 16].

For integer spin $n \geqslant 3$ the source constraint of Berends *et al* [5, 6, 16] or the Bianchi identity of de Wit and Freedman [4] is precisely the zero double trace of $\Gamma^{(3)}$, namely

$$\Gamma^{(3)\lambda}{}_{\lambda}{}^{\rho}{}_{\rho\mu_2\ldots\mu_n} = 0 \tag{25}$$

which implies

$$\partial^\lambda W_{\lambda\mu_2\ldots\mu_n} - \tfrac{1}{2}(n-1) W^\lambda{}_{\lambda(\mu_3\ldots\mu_n,\mu_2)} = 0 \tag{26}$$

where $W_{\mu_1\ldots\mu_n}$ is the left-hand side of the field equation. Expanding (26) in terms of the potential $\phi_{\mu_1\ldots\mu_n}$ yields, for $n \geqslant 4$,

$$\phi^\lambda{}_\lambda{}^\rho{}_{\rho(\mu_5\ldots\mu_n,\mu_2\mu_3\mu_4)} = 0 \tag{27}$$

and the Poincaré lemma may be used [15] to give the zero double trace condition [4] for the potential $\phi_{\mu_1\ldots\mu_n}$, namely

$$\phi^\lambda{}_\lambda{}^\rho{}_{\rho\mu_5\ldots\mu_n} = 0. \tag{28}$$

With the potential obeying the symmetry (28) the source constraint is identically satisfied.

For fermionic spin $j \geqslant \frac{7}{2}$ the combination

$$\tfrac{4}{3}\gamma^\lambda \Gamma^{(2)}{}_\lambda{}^\rho{}_{\rho\mu_2\ldots\mu_n} + \tfrac{2}{3}\gamma^\kappa\gamma^\rho\gamma^\lambda \Gamma^{(2)}{}_{\lambda\rho\kappa\mu_2\ldots\mu_n} = 0 \tag{29}$$

yields the appropriate source constraint equation and also

$$\gamma^\rho \psi_\rho{}^\lambda{}_{\lambda(\mu_4\ldots\mu_n,\mu_2\mu_3)} = 0 \tag{30}$$

and hence the vanishing double trace for $\psi_{\mu_1\ldots\mu_n}$:

$$\gamma^\rho \psi_\rho{}^\lambda{}_{\lambda\mu_4\ldots\mu_n} = 0. \tag{31}$$

For spins 1 to $\frac{5}{2}$ the source constraints are equivalent to equations satisfied by the field strengths.

Summarising, we have obtained all the above standard Lagrangian results for arbitrary helicity as natural consequences of an analysis based on integration of the very simple field strength irreps of the Poincaré group.

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